Acknowledgment

The authors wish to thank J. K. Vogel, W. Schilz, J. Köhler, and J. Magarshack for valuable comments and discussions.

References

[1] W. A. Edson, "Noise in oscillators," *Proc. IRE*, vol. 48, pp. 1454–1466, Aug. 1960.
[2] K. Kurokawa, "Noise in synchronized oscillators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 234–240, Apr. 1968.
[3] ——, "Some basic characteristics of broadband negative resistance oscillator circuits," *Bell Syst. Tech. J.*, vol. 48, pp. 1937–1955, 1969.
[4] E. J. Shelton, Jr., "Stabilization of microwave oscillators," *IRE Trans. Electron Devices* (*Symposium Issue*), vol. ED-1, pp. 30–40, Dec. 1954.
[5] I. Goldstein, "Frequency stabilization of a microwave oscillator with an external cavity," *IRE Trans. Microwave Theory Tech.*, vol. MTT-5, pp. 57–62, Jan. 1957.
[6] J. R. Ashley and C. B. Searles, "Microwave oscillator noise reduction by a transmission stabilizing cavity," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Noise*), MTT-16, pp. 743–748, 1968.
[7] C. Müller, "Unterdrückung des Frequenz-Rauschens von Gunn-Oszillatoren mit Hilfe eines äusseren Resonators," *Frequenz*, vol. 23, pp. 364–368, 1969.
[8] K. Schünemann and B. Schiek, "A comparison between transmission and reaction cavity stabilized oscillators," *Electron. Lett.*, vol. 7, p. 618, Oct. 1971.
[9] B. D. H. Tellegen and A. G. van Nie, unpublished work, Eindhoven, The Netherlands, 1957.

Stability Criteria for Phase-Locked Oscillators

G. H. BERTIL HANSSON AND K. INGEMAR LUNDSTRÖM

Abstract—**Stability criteria for negative conductance oscillators or amplifiers are derived in terms of the total circuit admittance. A figure of merit for phase locking at small injected powers is derived. The influence of large injected signals is studied. The conclusions drawn from the calculations are in good qualitative agreement with experimental observations on phase-locked IMPATT-diode oscillators.**

I. Introduction

PHASE-LOCKED oscillators have been shown a large interest in recent years due to the possibility of decreasing the FM noise of solid-state oscillators by injection locking. The purpose of this paper is to derive some general stability criteria for amplifiers and phase-locked oscillators whose active element can be described as a negative conductance (or negative resistance). The analysis is similar to that used by Kurokawa [1] and Brackett [2], who considered a general circuit in contrast to Adler [3], who studied a simple single resonant circuit. The stability criteria for a phase-locked oscillator are derived in a different way and cast in a different form that we find convenient to use. The main difference is, however, that we use a general series expansion for the negative conductance in contrast with Kurokawa who used a first-order approximation [1, eq. (11)]. One of the results of our theory is the introduction of two border lines for stable locking [4], which are called the *boundary* and *locus* curve, respectively, using a notation introduced by Golay [5], who studied the stability of a regenerative oscillator. It is shown by experiments that these two curves have practical implications. By calculating the boundary and locus curves, hysteresis and jumps in output power can be predicted.

Manuscript received November 22, 1971; revised March 6, 1972.
The authors are with the Research Laboratory of Electronics III, Chalmers University of Technology, Gothenburg, Sweden.

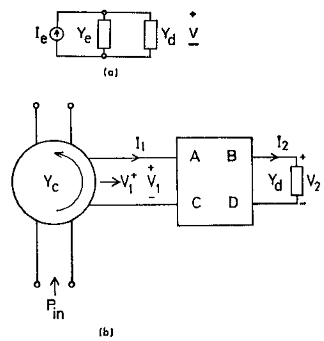

Fig. 1. (a) Equivalent circuit. (b) Circulator coupled negative conductance element.

The theory is applied to a simple cubic nonlinearity, with both a nonlinear conductance and susceptance. It is shown that the nonlinear susceptance introduces asymetrical locking properties at large injected powers.

II. Circuit Equations

The starting point for our calculations is the equivalent circuit shown in Fig. 1(a). In this circuit I_e is a current of frequency ω_i, which depends on the injected power P_{in}. Y_e is the admittance of the passive circuit as seen from the active element. The active element is described by a voltage-dependent susceptance

$$Y_d = G_d(V, \omega) + jB_d(V, \omega) \tag{1}$$

where V is the amplitude of the RF voltage across the active element. Y_e and I_e depend on the actual circuit. A circulator coupled negative conductance element, shown in Fig. 1(b), where the coupling circuit is described by its voltage–current transmission matrix

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{2}$$

has

$$Y_e = \frac{AY_c + C}{BY_c + D} \tag{3}$$

and

$$|I_e| = \left| \frac{Y_c}{BY_c + D} \right| \sqrt{8 P_{in}/G_c}. \tag{4}$$

$I_e = (2Y_c/(BY_c+D))V_1^+$, where V_1^+ is the incident voltage wave as shown in Fig. 1(b). The input power P_{in} is $|V_1^+|^2 G_c/2$. $Y_c \equiv G_c + jB_c$ is the characteristic admittance of the circulator.

If we introduce

$$Y(V, \omega) = Y_e(\omega) + Y_d(V, \omega) \tag{5}$$

we find that free-running oscillations require, with $Y \equiv G + jB$,

$$G(V_0, \omega_0) = 0 \tag{6}$$

and

$$B(V_0, \omega_0) = 0 \tag{7}$$

which defines the free-running amplitude V_0 and oscillation frequency ω_0. Generally, we have

$$Ie^{j(\omega_i t - \phi)} = Y(V, \omega) V e^{j\omega t} \tag{8}$$

where

$$I_e \equiv Ie^{-j\phi}$$

was introduced. I is the amplitude of I_e and ϕ the phase difference between I_e and the RF voltage V across the active element. ω is the instantaneous frequency of the RF voltage across the active element. In a stable locked state (or for a stable amplifier) $\omega = \omega_i$ and ϕ is independent of time. In this case we find from (8) by splitting it into real and imaginary parts

$$I \cos \phi = G(V, \omega_i) V \tag{9}$$

$$-I \sin \phi = B(V, \omega_i) V. \tag{10}$$

Squaring and adding (9) and (10) yields

$$I^2 = (G^2 + B^2) V^2. \tag{11}$$

Equations (9)–(11) thus describe the locked state and from them $V(I, \omega_i)$ and $\phi(I, \omega_i)$ may be computed. The question is, however, does a certain operating point represent a *stable locked state*?

III. Stability Criteria

An unlocked condition can be characterized by an RF voltage V of frequency ω_i with a time varying amplitude and phase. If $(1/V)(\partial V/\partial t)$ and $\partial\phi/\partial t$ are small quantities compared to ω_i, we may replace $Y(V, \omega)$ by [1].

$$Y(V, \omega) \approx Y(V, \omega_i) + \frac{\partial Y}{\partial \omega}\left(\frac{\partial \phi}{\partial t} - j\frac{1}{V}\frac{\partial V}{\partial t}\right) \tag{12}$$

where $\partial Y/\partial\omega$ is to be taken in the point (V, ω_i). Equation (12) means that we have replaced an RF voltage of the form $V(t)e^{j(\omega_i t + \phi(t))}$ by $V(t)e^{j(\omega t + \phi')}$, where $\omega = \omega_i + (\partial\phi/\partial t)|_{t'} - j(1/V)(\partial V/\partial t)|_{t'}$ and where t' is a given time and ϕ' a fixed phase angle. ω is the instantaneous frequency, and if $\omega - \omega_i \ll \omega_i$, a series expansion of Y around (V, ω_i) may be made as indicated in (12). A more rigorous derivation of (12) is given in [1]. If we use this Y in (8) and split (8) into real and imaginary parts we find

$$I \cos \phi = \left[G(V, \omega_i) + \frac{\partial G}{\partial \omega}\frac{\partial \phi}{\partial t} + \frac{\partial B}{\partial \omega}\frac{1}{V}\frac{\partial V}{\partial t}\right] V \tag{13}$$

$$-I \sin \phi = \left[B(V, \omega_i) + \frac{\partial B}{\partial \omega}\frac{\partial \phi}{\partial t} - \frac{\partial G}{\partial \omega}\frac{1}{V}\frac{\partial V}{\partial t}\right] V \tag{14}$$

where the derivatives with respect to frequency are taken in (V, ω_i). For a stable locked oscillator, of course $(\partial\phi/\partial t) = (\partial V/\partial t) = 0$ and (13) and (14) reduce to (9) and (10). To determine if V and ϕ, derived from (9) and (10), represent a stable locking point we perturb V and ϕ by small amounts ΔV and $\Delta\phi$ and see if they will grow or decay with time. Replacing $\partial/\partial t$ by s we find from (13) and (14), to first order in ΔV and $\Delta\phi$:

$$-\Delta\phi I \sin \phi = \left[G(V, \omega_i) + \frac{\partial G}{\partial V} V + \frac{\partial B}{\partial \omega} s\right] \Delta V + \frac{\partial G}{\partial \omega} V s \Delta\phi \tag{15}$$

$$-\Delta\phi I \cos \phi = \left[B(V, \omega_i) + \frac{\partial B}{\partial V} V - \frac{\partial G}{\partial \omega} s\right] \Delta V + \frac{\partial B}{\partial \omega} V s \Delta\phi. \tag{16}$$

Using (9) and (10) we find

$$\left(G + \frac{\partial G}{\partial V} V + s\frac{\partial B}{\partial \omega}\right)\frac{\Delta V}{V} - \left(B - s\frac{\partial G}{\partial \omega}\right)\Delta\phi = 0 \tag{17}$$

$$\left(B + \frac{\partial B}{\partial V} V - s\frac{\partial G}{\partial \omega}\right)\frac{\Delta V}{V} + \left(G + s\frac{\partial B}{\partial \omega}\right)\Delta\phi = 0. \tag{18}$$

The stability of the locked oscillator is determined by the zeros of the determinant to (17) and (18). Stability requires that all roots of s lie in the left-hand part of the complex plane, which requires

$$2\left(G\frac{\partial B}{\partial \omega} - B\frac{\partial G}{\partial \omega}\right) + \left(\frac{\partial G}{\partial V}\frac{\partial B}{\partial \omega} - \frac{\partial B}{\partial V}\frac{\partial G}{\partial \omega}\right) V > 0 \tag{19}$$

and

$$G\left(G + \frac{\partial G}{\partial V} V\right) + B\left(B + \frac{\partial B}{\partial V} V\right) > 0. \tag{20}$$

The curves given by *equal signs* in (19) and (20) are named *boundary curve* and *locus curve*, respectively.

Equation (20) is equivalent to $(\partial I^2/\partial V)>0$, which is easily seen from (13), which means that for a stable locked oscillator we require the RF voltage across the active element to increase if the injected power ($P_{in}\sim I^2$) increases. It may be noted that for a free-running oscillator the stability criterion (20) disappears and (19) reduces to

$$\frac{\partial G}{\partial V}\frac{\partial B}{\partial \omega}-\frac{\partial B}{\partial V}\frac{\partial G}{\partial \omega}>0.$$

The expression on the left-hand side is a measure of the dynamic stability of the free-running oscillator. The normalized dynamic stability factor

$$S=\frac{\dfrac{\partial G}{\partial V}\dfrac{\partial B}{\partial \omega}-\dfrac{\partial B}{\partial V}\dfrac{\partial G}{\partial \omega}}{\sqrt{\left(\left(\dfrac{\partial G}{\partial V}\right)^2+\left(\dfrac{\partial B}{\partial V}\right)^2\right)\left(\left(\dfrac{\partial G}{\partial \omega}\right)^2+\left(\dfrac{\partial B}{\partial \omega}\right)^2\right)}} \tag{21}$$

has a maximum value of 1.0, which would characterize an oscillator dynamically stabilized in an optimum way. The normalized dynamic stability is a measure of the relative stability that appears in calculations of the oscillator noise and AM to FM conversion [1].

For small injected power and small frequency deviations, $|\omega_i-\omega_0|$, the following approximation can be used

$$G+jB=\frac{\partial G}{\partial V}(V-V_0)+\frac{\partial G}{\partial \omega}(\omega_i-\omega_0)+j\frac{\partial B}{\partial V}(V-V_0)+j\frac{\partial B}{\partial \omega}(\omega_i-\omega_0) \tag{22}$$

where the derivatives are taken in the point (V_0,ω_0). From (20) we get

$$\left\{\frac{\partial G}{\partial V}(V-V_0)+\frac{\partial G}{\partial \omega}(\omega_i-\omega_0)\right\}\frac{\partial G}{\partial V}V_0+\left\{\frac{\partial B}{\partial V}(V-V_0)+\frac{\partial B}{\partial \omega}(\omega_i-\omega_0)\right\}\frac{\partial B}{\partial V}V_0=0 \tag{23}$$

where second-order terms in $(V-V_0)$ and $(\omega_i-\omega_0)$ have been neglected. This gives an expression for the voltage on the locus curve

$$V-V_0=-\frac{\dfrac{\partial G}{\partial V}\dfrac{\partial G}{\partial \omega}+\dfrac{\partial B}{\partial V}\dfrac{\partial B}{\partial \omega}}{\left(\dfrac{\partial G}{\partial V}\right)^2+\left(\dfrac{\partial B}{\partial V}\right)^2}\cdot(\omega_i-\omega_0) \tag{24}$$

Introducing this in (11) gives

$$\frac{I^2}{V_0^2}=\frac{\left(\dfrac{\partial G}{\partial V}\dfrac{\partial B}{\partial \omega}-\dfrac{\partial B}{\partial V}\dfrac{\partial G}{\partial \omega}\right)^2}{\left(\dfrac{\partial G}{\partial V}\right)^2+\left(\dfrac{\partial B}{\partial V}\right)^2}(\omega_i-\omega_0)^2. \tag{25}$$

Since the locus curve defines the stability region of the locked oscillator (at small injected powers) (25) gives the maximum locking range $(\omega_i-\omega_0)_{max}$ for a certain injected power. Replacing I and V_0 by input power P_{in} and free-running output power P_o we get

$$\left|\frac{\omega_i-\omega_0}{\omega_0}\right|_{max}=\frac{1}{Q_{ext}}\sqrt{\frac{P_{in}}{P_o}} \tag{26}$$

where

$$Q_{ext}=\frac{\omega_0}{2G_c}\frac{\left|\dfrac{\partial G}{\partial V}\dfrac{\partial B}{\partial \omega}-\dfrac{\partial B}{\partial V}\dfrac{\partial G}{\partial \omega}\right|}{\sqrt{\left(\dfrac{\partial G}{\partial V}\right)^2+\left(\dfrac{\partial B}{\partial V}\right)^2}}=\frac{\omega_0}{2G_c}|S|\sqrt{\left(\frac{\partial B}{\partial \omega}\right)^2+\left(\frac{\partial G}{\partial \omega}\right)^2}. \tag{27}$$

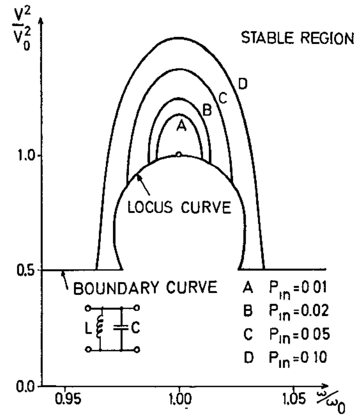

Fig. 2. Normalized voltage versus frequency for different input powers. The coupling circuit has $L=0.1$ and $C=10.0$. Furthermore, $G_c=1.0$, $Y_d=-2.0+G_2V^2(C_2=0)$. P_{in} is the injected power normalized by the free-running output power, $P_o=(G_cV_0^2)/2$.

This result should be compared with Adler's [3]. Equation (26) has the same form as Adler's but the "normal" $Q_{ext}=(\omega_0/2G_C)(\partial B/\partial\omega)=(\omega_{0c}/G_c)$ has to be replaced by (27). Equation (19) in [1] can be cast in the same form as (26) and (27).

Although we have discussed a phase-locked oscillator, the derived stability criteria are of course also applicable to a negative conductance amplifier. We should remember, however, that the boundary curve, defined by (19), is accurate only as long as (12) can be used.

IV. Example

The theory has been applied to a simple cubic nonlinearity provided with different coupling circuits [4]. By using the stability criteria a number of experimentally found phenomena could be explained, e.g., hysteresis and jumps in output power and unsymmetrical locking ranges [4]. As an example we will study a parallel tuned circuit. The active element is described by the following admittance

$$Y_d=G_0+G_2V^2+j\omega C_0+j\omega C_2V^2. \tag{28}$$

Fig. 2 shows V^2/V_0^2 versus ω/ω_0 for different input

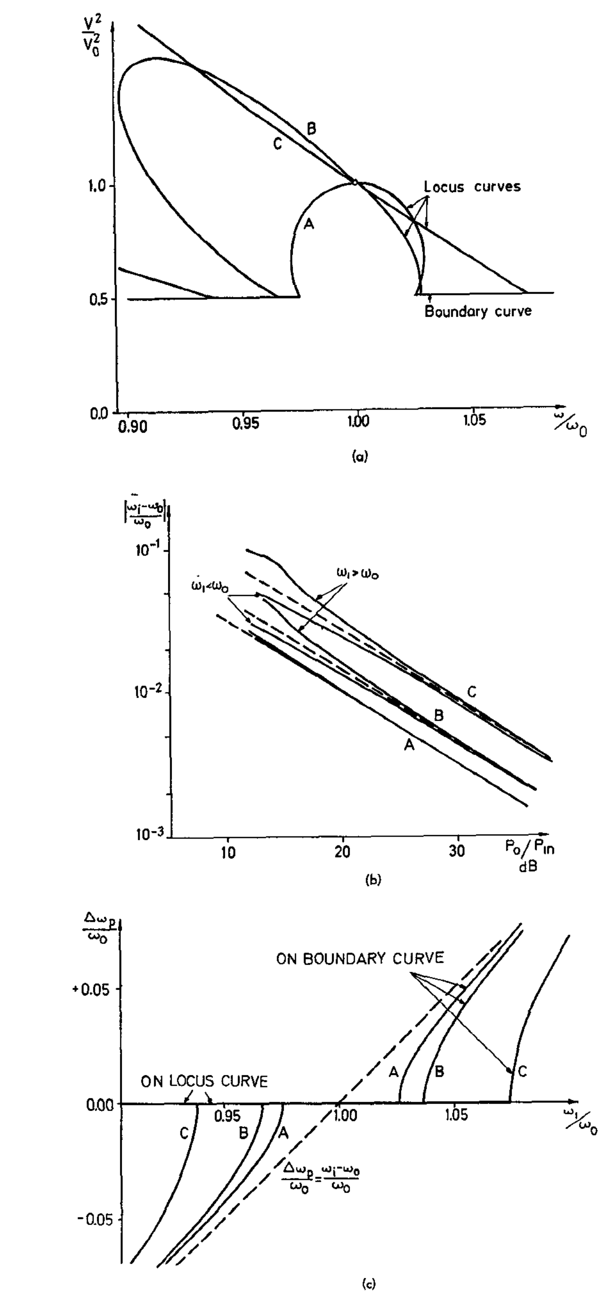

Fig. 3. (a) Locus and boundary curves. (b) Locking bandwidth versus locking gain. (c) Modulation frequency versus frequency on the locus and boundary curves. $G_c = 1.0$, $L = 0.1$, $G_0 = -2.0$; A: $B_2/G_2 = 0$ and $C = 10.0$; B: $B_2/G_2 = 1.0$ and $C = 9.0$; C: $B_2/G_2 = 2.5$ and $C = 7.5$.

powers. V_0 and ω_0 are given by (6) and (7). The parameter values chosen are shown in the figure captions. The region of stable locking as calculated from the stability criteria is shown.

P_{in} is the injected power normalized by the free-running output power $P_o = (G_c V_0^2)/2$, and since we have plotted V^2/V_0^2 the value of G_2 is arbitrary and incorporated in V_0^2.

In Fig. 3(a) locus and boundary curves are shown for an active element with the same coupling circuit as in Fig. 2, but with different values of the voltage-dependent capacitance. The circuit is tuned to give a

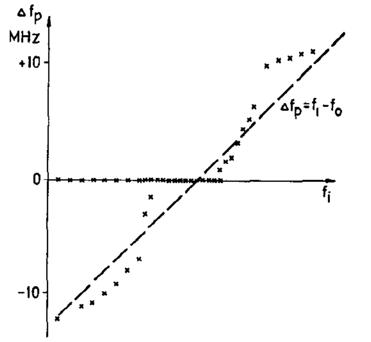

Fig. 4. Experimental values of modulation frequency versus frequency on the border of the stable region. $f_0 = 9975$ MHz.

constant ω_0. The slope of the locus curve at ω_0 is given by (24). The slope has a maximum for $B_2 = G_2$. Fig. 3(a) shows that unsymmetrical locking properties are obtained due to the voltage-dependent susceptance. This is also shown in Fig. 3(b), which is the common locking bandwidth versus gain diagram. The dashed lines are given by (26) with Q_{ext} computed from (27). This expression is, however, only valid at small P_{in} (large P_o/P_{in}). At larger P_{in} the asymmetry of the locus curve makes the locking bandwidth larger for positive frequency deviations (in this example).

By actually calculating the roots of s from the determinental equation on the border of the stable region some information may be gained concerning the type of instability which is the most important at different parts of the border. Close to locking the output is a carrier with amplitude and phase modulation. In Fig. 3(c) the modulation frequency $\Delta\omega_p$ is shown. The modulation frequency $\Delta\omega_p$ is the difference between the frequency of the injected signal ω_i and the frequency of the oscillator at the border of stable locking.

On the locus curve the modulation frequency is zero and on the boundary curve the modulation frequency goes toward the frequency difference $\omega_i - \omega_0$. This means that when the stable locking range is determined by the locus curve we have complete frequency pulling, i.e., $\Delta\omega_p = 0$. When the stable locking region is determined by the boundary curve, the frequency of the oscillator is only partly pulled (or in some cases pushed from) the injected signal at the border of locking. For injected signals far away from the free-running frequency ω_0 we have neither frequency pushing nor pulling, which means that the modulation frequency $\Delta\omega_p$ equals $\omega_i - \omega_0$.

This is in good agreement qualitatively with experiments done on an IMPATT-diode oscillator with $f_0 = 9975$ MHz and $P_o = 60$ mW and also with conclusions made using a different method of analyzing nonlinear microwave circuits [6].

The experimental results are shown in Fig. 4. The crosses in Fig. 4 represent the limit of the modulation frequency obtained at a certain injected frequency, just at the border of locking. They were obtained by choosing a frequency f_i and increasing P_i until locking occurred. Points lying on the f_i axis represent the modulation frequency obtained when the locking range is determined by the locus curve. The other points represent the modulation frequency limit when the locking range is determined by the boundary curve. For some injected frequencies ($f_i < f_0$) there are two crosses at the same frequency. The reason for this may be seen in Fig. 3(a). We look at case B with $\omega_i/\omega_0 = 0.95$ as an example. Increasing the input power from zero we find that locking occurs at the boundary curve and hence gives a cross close to $f_i - f_0$ in Fig. 4. If the input power increased further (which for a stable locked state also means increasing the voltage amplitude V) we reach the locus curve B and a jump in V (and output power) occurs. The modulation frequency just before the jump is then zero. Furthermore, this example shows the possibility of hysteresis and jumps in the output power of a locked oscillator.

V. Conclusions

Some general stability criteria have been derived that seem to be useful in connection with practical amplifiers and oscillators. The results obtained are in good agreement qualitatively with experimental results. Furthermore, a new locking figure of merit $1/Q_{ext}$ was derived for small injected powers. It is seen from this figure of merit, (27), that introducing a voltage-dependent capacitance may increase the locking bandwidth. However, this also decreases the dynamic stability of the oscillator. To increase the locking bandwidth without decreasing the dynamic stability one has to decrease the absolute value of the frequency derivative of the admittance. Furthermore, the modulation of the oscillator at the border to phase locking was studied. The difference between locking at the boundary and locus curve, respectively, was pointed out. The effect of large injected power was shown to create unsymmetric locking properties if a nonlinear susceptance is present in the circuit.

References

[1] K. Kurokawa, "Some basic characteristics of broadband negative resistance oscillator circuits," *Bell Syst. Tech. J.*, vol. 48, pp. 1937–1955, 1969.
[2] C. A. Brackett, "Characterization of second-harmonic effects in IMPATT diodes," *Bell Syst. Tech. J.*, vol. 49, pp. 1777–1810, 1970.
[3] R. Adler, "A study of locking phenomena in oscillators," *Proc. IRE*, vol. 34, pp. 351–357, June 1946.
[4] B. Hansson and I. Lundström, "Phase locking of negative conductance oscillators," in *Proc. 1971 European Microwave Conf.*, pp. A6/4:1–4.
[5] M. J. E. Golay, "Normalized equations of the regenerative oscillator—Noise, phase-locking, and pulling," *Proc. IEEE*, vol. 52, pp. 1311–1330, Nov. 1964.
[6] L. Gustafsson, G. H. B. Hansson, and K. I. Lundström, "On the use of describing functions in the study of nonlinear active microwave circuits," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 402–409, June 1972.

A Wide-Band Gunn-Effect CW Waveguide Amplifier

AMADOU SÉNE AND FRED J. ROSENBAUM

***Abstract*—Broad-band CW amplification with Gunn diodes in waveguide circuits has been obtained, with power gains typically between 10 and 15 dB and half-power bandwidths of more than 1 GHz. It is found that amplifier performance can be modeled with fair accuracy using a rough characterization for the diode parameters.**

Introduction

Although Gunn diodes have been used primarily in oscillator applications, their negative resistance properties can also be used to obtain reflection gain [1]–[8]. Three amplifying modes have been observed that depend on the product of doping density n and the length L of the GaAs chip.

Manuscript received October 14, 1971; revised April 16, 1972. This work was supported in part by the African-American Institute and in part by NASA under Grant Y/NGL-26-008-054.
A. Séne was with the Electrical Engineering Department, Washington University, St. Louis, Mo. 63130. He is now with Quakam-Tak Pou, Dakar-Banlieue, Senegal.
F. J. Rosenbaum is with the Electrical Engineering Department, Washington University, St. Louis, Mo. 63130.

McCumber and Chynoweth [9] have shown theoretically that when the nL product is less than 5×10^{11} cm^{-2} a Gunn diode exhibits a negative resistance around the transit-time frequency and its harmonics [1]. Such diodes are usually referred to as being subcritically doped, since when biased above threshold they do not enter into transit-time (Gunn) oscillations. The main disadvantage of subcritically doped amplifiers is the high ratio of capacitance-to-negative conductance that allows circuit matching for high gain only in a narrow frequency band.

When the nL product is above 5×10^{11} cm^{-2} the device is supercritically doped. If the bias voltage is just above threshold, high field domains nucleate near the cathode boundary and the diode may oscillate in the